CHAPTER 3

Progression

3.1 INTRODUCTION

A production manager of an enterprise spends Rs. 65,000 for production every year. He proposes to spend Rs. 15,000 more every year to meet the increasing demand of the market. If he wants to find how much he has to spend after 10 years or the sum of all expenditure during 10 years, he can use arithmetic progression.

In this chapter we will discuss about arithmetic, geometric and harmonic progressions. These are widely used to find the compound interest earned, sum on recurring deposits, depreciation after certain time, etc.

3.2 SEQUENCE

In daily life we come across the arrangement of certain things in an order. Such as arrangement of students in a row as per their roll numbers, arrangement of books in the library etc.

A sequence is an ordered arrangement of numbers according to a given rule. The individual numbers that form a sequence are the terms of a sequence.

Example: 5, 10, 15, 20, ... form a sequence.

5, 10, 15, 20 are called first, second, 3rd, 4th ... terms of the sequence. They are denoted by $T_1, T_2, T_3, T_4, \ldots$ respectively.

3.3 ARITHMETIC PROGRESSION

A sequence in which the difference between any term and its previous term is constant is called arithmetic progression (A.P.). The constant is called common difference.

Example: 0, 2, 4, 6, 8, ... is an A.P. with first term 0 and common difference = 2.

General term or n^{th} term of an arithmetic progression

Let 'a' be the first term and 'd' be the common difference of an A.P.

First term $= T_1 = a$

Second term $= T_2 = a + d$ $\qquad (= a + (2 - 1)\, d)$

Third term $= T_3 = a + 2d$ $(= a + (3 - 1)\,d)$

Fourth term $= T_4 = a + 3d$ $(= a + (4 - 1)\,d)$

⋮ ⋮ ⋮ ⋮

n^{th} term $= T_n = a + (n - 1)\,d$

n^{th} term of an A.P. whose first term is a and common difference is d is $a + (n - 1)\,d$.

Note: If 'a' and 'd' are known, we can write all the terms of A.P. as a, $a + d$, $a + 2d$, $a + 3d$, ...

Sum of first 'n' terms of an arithmetic progression

Let 'a' be the first term 'd' be the common difference and S_n be the sum of first 'n' terms of an arithmetic progression.

$$S_n = a + a + d + a + 2d + \ldots + a + (n - 1)\,d$$

Let

$$\ell = a + (n - 1)\,d = T_n$$

Then

$$S_n = a + a + d + a + 2d + \ldots \ell$$

Since the preceeding term to $a + d$ is a. The preceeding term to ℓ is $\ell - d$. The preceeding term to $\ell - d$ is $\ell - 2d$.

$$S_n = a + a + d + a + 2d + \ldots \ell - 2d + \ell - d + \ell \quad \ldots(1)$$

Writing in reverse order

$$S_n = \ell + \ell - d + \ell - 2d + \ldots a + 2d + a + d + a \quad \ldots(2)$$

Adding (1) and (2) we get

$$2S_n = (a + \ell) + (a + \ell) \ldots n \text{ times}$$

$$2S_n = n\,(a + \ell)$$

$$S_n = \frac{n}{2}(a + \ell)$$

$$S_n = \frac{n}{2}[a + a + (n - 1)\,d] \qquad \because \ell = a + (n - 1)\,d$$

$$\therefore S_n = \frac{n}{2}[2a + (n - 1)\,d]$$

Sum of the first 'n' terms of an AP.

$$= \frac{n}{2}[2a + (n - 1)\,d]$$

If T_n is the n^{th} term. Then $S_n = \frac{n}{2}[a + T_n]$

3.4 ARITHMETIC MEANS

If a, x_1, x_2 x_3, ... x_n, b are in arithmetic progression, then $x_1, x_2, \ldots x_n$ are called 'n' arithmetic means between a and b.

To find the arithmetic means we note that $a, x_1, x_2, \ldots, x_n, b$ are $(n + 2)$ terms of A.P. whose first term is a and $(n + 2)^{nd}$ term i.e. $T_{n+2} = b$.

But $\quad T_{n+2} = a + ((n + 2) - 1)\, d$

$$b = a + (n + 1)\, d$$

$\Rightarrow \quad b - a = (n + 1)\, d$

$\Rightarrow \quad d = \dfrac{b-a}{n+1}$

After finding d we can get

$$x_1 = a + d$$

$$x_2 = a + 2d \;\ldots\ldots\; x_n = a + nd.$$

Arithmetic mean: If a, x, b are in arithmetic progression then x is called arithmetic mean between a and b.

Since a, x, b are in arithmetic progression,
any term – previous term = constant.

$\therefore \quad x - a = b - x$

$$x + x = b + a$$

$$2x = a + b$$

$$x = \frac{a+b}{2}.$$

$\therefore$ The arithmetic mean between a and b is $\dfrac{a+b}{2}$

WORKED EXAMPLES

1. Find the 26^{th} term of an A.P. 9, 6, 3, 0, –3, ...

Solution: $\quad a = 9;\ d = 6 - 9 = 3 - 6 = -3$

$n = 26$

$T_{26} = ?$

$$T_n = a + (n - 1)\, d \text{ (formula)}$$

$$T_{26} = 9 + (26 - 1)(-3)$$

$$T_{26} = 9 + 25 \times (-3) = -66$$

2. What term of A.P. 2, 5, 8, ... is 56?

Given:

$$a = 2$$

$$d = 5 - 2 = 3$$
$$T_n = 56$$
$$n = ?$$
$$T_n = a + (n - 1)\, d \text{ (formula)}$$
$$56 = 2 + (n - 1)\, 3$$
$$56 - 2 = (n - 1)\, 3$$
$$54 = (n - 1)\, 3$$
$$\frac{54}{3} = n - 1$$
$$18 = n - 1$$
$$n = 18 + 1 = 19$$
$$n = 19$$

3. Find the sum of first 18 terms of an A.P. $5\frac{1}{2}, 6\frac{3}{4}, 8, \ldots\ldots$

Given:

$$a = 5\frac{1}{2} = \frac{11}{2}$$
$$d = 6\frac{3}{4} - 5\frac{1}{2} = 1\frac{1}{4} = \frac{5}{4}$$
$$n = 18$$
$$S_n = ?$$
$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$S_n = \frac{18}{2}\left[2 \cdot \frac{11}{2} + (18-1) \cdot \frac{5}{4}\right]$$
$$S_n = 9\left[11 + \frac{85}{4}\right]$$
$$S_n = 9\left[\frac{44+85}{4}\right] = 9\left[\frac{129}{4}\right]$$
$$S_n = \frac{1161}{4} = 290\frac{1}{4}$$

4. If the 8^{th} term and 20^{th} term of an A.P. are 22 and 46 respectively. Find the A.P. and hence find its 18^{th} term.

Given $T_8 = 22$

$T_{20} = 46$

$T_{18} = ?$

Let a be the first term and 'd' be the common difference.

$T_n = a + (n - 1)\, d$

We have $T_8 = a + (8 - 1)\, d = 22$

$a + 7d = 22$ ...(1)

$T_{20} = a + (20 - 1)\, d = 46$

$a + 19d = 46$ ...(2)

Solving (1) and (2)

$$\begin{array}{rcrcr} a & + & 7d & = & 22 \\ a & + & 19d & = & 46 \\ (-) & & (-) & & (-) \\ \hline & & -12d & = & -24 \end{array}$$

$\Rightarrow d = 2$

Substituting $d = 2$ in (1)

$a + 7d = 22$

$a + 7(2) = 22$

$a = 22 - 14$

$a = 8$

$\therefore$ Required A.P. = $a, a + d, a + 2d, a + 3d, \ldots$

i.e. $8, 8 + 2, 8 + 2(2) + 8 + 3(2) + \ldots$

$8, 10, 12, 14, \ldots$

18^{th} term $= T_{18} = a + (18 - 1)\, d$

$= 8 + (18 - 1)\, 2$

$= 8 + 17\, (2)$

$= 42.$

5. The first and last terms of A.P. are 2 and 202. If the A.P. has 30 terms, then find the sum of these terms.

Given:

$a = 2,$

$\ell = 202$

$n = 30$

$S_n = ?$

$$S_n = \frac{n}{2}[a+l] \qquad \text{(formula)}$$

$$S_{30} = \frac{30}{2}[2+202]$$

$$S_{30} = 15[204] = 3060$$

6. How many terms of the series 1 + 5 + 9 + ... must be taken to give a sum of 2415.

Solution: The terms 1, 5, 9, ... are in A.P. with $a = 1$, $d = 5 - 1 = 9 - 5 = 4$.

$$S_n = 2415$$

$$n = ?$$

$$S_n = \frac{n}{2}[2a+(n-1)d] \qquad \text{(formula)}$$

$$2415 = \frac{n}{2}[2(1)+(n-1)4]$$

$$2415 \times 2 = n[2+4n-4]$$

$$4830 = n[4n-2]$$

$$4830 = 4n^2 - 2n \div \text{by } 2 \qquad 2 \times 2415 \begin{array}{l} -70 \\ +69 \\ \hline -1 \end{array}$$

$$2415 = 2n^2 - n$$

$$2n^2 - n - 2415 = 0$$

$$2n^2 - 70n + 69n - 2415 = 0$$

$$2n(n-35) + 6n(n-35) = 0$$

$$n = 35 \text{ or } n = -\frac{69}{2}$$

$\because$ n is non-negative.

$$n = 35.$$

7. Insert 6 arithmetic means between 1 and 19.

Solution: Let 1, x_1, x_2, x_3, x_4, x_5, x_6, 19 be the required arithmetic progression.

$a = 1$, $T_8 \triangleq 8^{th}$ term $= 19$

$$n^{th} \text{ term} = T_n = a+(n-1)d \text{ (formula)}$$

But

$$19 = 1+(8-1)\,d$$

$$19 = 1 + 7d$$

$$19 - 1 = 7d$$

$$\Rightarrow \quad d = \frac{18}{7}$$

Hence

$$x_1 = a + d = 1+\frac{18}{7} = \frac{25}{7}$$

$$x_2 = a + 2d = x_1 + d = \frac{25}{7}+\frac{18}{7} = \frac{43}{7}$$

$$x_3 = x_2 + d = \frac{43}{7}+\frac{18}{7} = \frac{61}{7}$$

$$x_4 = x_3 + d = \frac{61}{7}+\frac{18}{7} = \frac{79}{7}$$

$$x_5 = x_4 + d = \frac{79}{7}+\frac{18}{7} = \frac{97}{7}$$

$$x_6 = x_5 + d = \frac{97}{7}+\frac{18}{7} = \frac{115}{7}$$

So, the required means are

$$\frac{25}{7}, \frac{43}{7}, \frac{61}{7}, \frac{79}{7}, \frac{97}{7}, \frac{115}{7}$$

8. There are n arithmetic means between 3 and 35. So that the ratio of 4^{th} mean to n^{th} mean is 1:3. Find n.

Solution: Let 3, x_1, x_2, x_3, ... x_n. 35 be the corresponding arithmetic progression.

4^{th} mean $= x_4 = 5^{th}$ term of the A.P. $= a + (5-1)\,d = 3 + 4d$

n^{th} mean $= x_n =$ Last but one term of the A.P

$= T_n - d = 35 - d$

Given

$$\frac{4^{th} \text{ mean}}{n^{th} \text{ mean}} = \frac{1}{3}$$

$$\frac{3+4d}{35-d} = \frac{1}{3}$$

Cross multiplying

$$3(3 + 4d) = 1(35 - d)$$
$$9 + 12d = 35 - d$$
$$9 - 35 = -12d - d$$
$$-26 = -13d$$
$$\Rightarrow \quad d = 2$$
$$(n + 2)^{nd} \text{ terms} = 35$$
$$a + (n + 2 - 1)\, d = 35$$
$$3 + (n + 1)\, 2 = 35$$
$$\Rightarrow (n + 1)\, 2 = 35 - 3$$
$$\Rightarrow (n + 1)\, 2 = 32$$
$$\Rightarrow \quad n + 1 = 16$$
$$\Rightarrow \quad n = 15$$

9. If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P. Then prove that a^2, b^2, c^2 are also in A.P.

Given: $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

So, any term – previous term = Constant

$$\Rightarrow T_2 - T_1 = T_3 - T_2$$

$$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

$$\frac{(b+c)-(c+a)}{(c+a)(b+c)} = \frac{(c-a)-(a+b)}{(a+b)\,(c+a)}$$

$$\frac{b+c-c-a}{b+c} = \frac{c+a-a-b}{a+b}$$

$$\frac{b-a}{b+c} = \frac{c-b}{a+b}$$

Cross multiplying

$$(b-a)(a+b) = (b+c)(c-b)$$
$$b^2 - a^2 = c^2 - b^2$$

$\Rightarrow$ a^2, b^2, c^2 are in A.P.

Hence proved.

10. The sum of 3 number in A.P. is 21 and their product is 280. Find the numbers.

Solution: Let the 3 numbers in A.P. be $x - d, x, x + d$.

Given their sum = 21

$$x - d + x + x + d = 21$$

$$3x = 21$$

$$x = 7$$

$\Rightarrow$ Their product = 280

$$(x-d)x(x+d) = 280$$

$$(x^2 - d^2)x = 280$$

$$(7^2 - d^2)7 = 280$$

$$49 - d^2 = 40^2$$

$$49 - 40 = d^2$$

$\Rightarrow$ $d = 3$

$\therefore$ The number are $7-3, 7$ and $7+3$ *i.e.*, 4, 7, 10.

11. The sum of 4 numbers in A.P. is 32 and product of the extremes numbers is 55. Find the numbers.

Solution: Let the 4 numbers in A.P. be

$$x - 3d, x - d, x + d, x + 3d$$

Given: Their sum = 32

$$x - 3d + x - d + x + d + x - 3d = 32$$

$$4x = 32$$

$$x = 8$$

Product of extreme numbers = 55

$$(x-3d)(x+3d) = 55$$

$$x^2 - 9d^2 = 55$$

$$64 - 9d^2 = 55$$

$$64 - 55 = 9d^2$$

$$9 = 9d^2$$

$$d = 1$$

$\therefore$ The numbers are 8 – 3, 8 – 1, 8 + 1, 8 + 3

5, 7, 9, 11

12. If the p^{th} term of an A.P. is q and q^{th} term is p. Then prove that $(p + q)^{th}$ term is 0.

Let a be the first term and d be the common difference of A.P.

Proof: Given p^{th} term = q

$$T_p = q$$

$$T_n = a + (n-1)\,d \qquad \text{[Formula]}$$

But

$$T_p = a + (p-1)\,d = q \qquad \ldots(1)$$

$\therefore$

Also Given

$$T_q = p$$

$$T_q = a + (q-1)\,d = p \qquad \ldots(2)$$

To prove:

$$T_{p+q} = 0$$

i.e.,

$$a + (p+q-1)\,d = 0 \qquad \ldots(3)$$

Solving (1) and (2) to get value of a and d:

$$\begin{array}{rcl} a + (p-1)d &=& q \\ a + (q-1)d &=& p \\ (-) \quad (-) & & (-) \\ \hline (p-1-q+1)d &=& q-p \end{array}$$

$$(p-q)\,d = q-p$$

$$d = \frac{q-p}{p-q} = \frac{q-p}{-(q-p)}$$

$$d = -1$$

Substituting $d = -1$ in (1)

$$a + (p-1)(-1) = q$$

$$a = q + p - 1$$

Consider LHS of equation (3)

$$a + (p+q-1)\,d$$

Substituting $a = p + q - 1$ and $d = -1$

we get

$$(p+q-1) + (p+q-1)(-1)$$

$$p + q - 1 - p - q + 1 = 0 = \text{RHS}$$

Hence proved.

13. Find the sum of all integers between 300 and 800 which are divisible by 9.

Solution: First number above 300 that is divisible by 9 is 306 and last number below 800 that is divisible by 9 is 792.

We need

$$306 + 315 + \ldots + 792$$

This is an A.P. with $a = 306$, $d = 9$.

$$T_n = 792$$

$$T_n = a + (n-1)\,d$$

But

$$792 = 306 + (n-1)\,9$$

$$792 - 306 = (n-1)\,9$$

$$486 = (n-1)\,9$$

$$\Rightarrow \quad n - 1 = \frac{486}{9}$$

$$n - 1 = 54$$

$$n = 54 + 1 = 55$$

Now,

$$S_n = \frac{n}{2}[a + T_n]$$

$$S_{55} = \frac{55}{2}[306 + 792]$$

$$S_{55} = 30{,}195.$$

14. If S_1, S_2, S_3 are respectively the sum of the first n, $2n$ and $3n$ terms of A.P. Then prove that $S_3 = 3\,(S_2 - S_1)$

Proof: Let a be the first term and 'd' be the common difference.

We have sum of first n terms of an A.P.

$$= S_n = \frac{n}{2}[2a + (n-1)d]$$

Given S_1 = Sum of first n terms $= \frac{n}{2}[2a + (n-1)d]$

S_2 = Sum of first 2n terms $= \frac{2n}{2}[2a + (2n-1)d]$

S_3 = Sum of first 3n terms $= \frac{3n}{2}[2a + (3n-1)d]$

To prove: $S_3 = 3\,(S_2 - S_1)$

Consider RHS: $3\,(S_2 - S_1)$

$$= 3\left[\frac{2n}{2}(2a + (2n-1)\,d) - \frac{n}{2}(2a + (n-1)\,d)\right]$$

$$= \frac{3n}{2}[4a + 2(2n-1)d - 2a - (n-1)d]$$

$$= \frac{3n}{2}[2a + (4n - 2 - n + 1)\,d]$$

$$= \frac{3n}{2}[2a + (3n-1)\,d] = S_3 = \text{L.H.S.}$$

Hence proved.

15. Prove that in an A.P. $T_{m+n} + T_{m-n}$ is independent of n.

Proof: Let a be the first term 'd' be the common difference of an A.P.

We have n^{th} term

$$T_n = a + (n-1)\,d$$

$$T_{m+n} = a + (m+n-1)d$$

$$\therefore \quad T_{m-n} = a + (m-n-1)d$$

Adding

$$T_{m+n} + T_{m-n} = a + (m-n-1)d + a + (m-n-1)d$$

$$= 2a + (m+n-1+m-n-1)d$$

$$= 2a + (2m-2)d$$

$$T_{m+n} + T_{m-n} = 2[a + (m-1)d] \text{ which is free from } n.$$

$\therefore$ It is independent of n.

3.5 APPLICATION OF ARITHMETIC PROGRESSION TO BUSINESS PROBLEMS

16. A man lends Rs. 2415 to a friend agreeing to charge no interest and also to recover the amount by daily installment increasing successfully by Rs. 4. In how many days will the loan be paid up if the first installment is Re. 1.

Solution: Let n be number of days.

$$S_n = \text{sum to } n \text{ terms} = 2415$$

This is an A.P. with $a = 1,\ d = 4,\ S_n = 2415$

Formula: $$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$2415 = \frac{n}{2}[2(1) + (n-1)4]$$

$$2415 = n + 2n^2 - 2n$$

$$2n^2 - n - 2415 = 0$$

$$2n^2 - 70n + 69n - 2415 = 0$$

$$2n(n-35) + 69(n-35) = 0$$

$$n = 35 \text{ or } n = -\frac{69}{2} \quad \because n \text{ is non-negative, } n = 35.$$

$\therefore$ In 35 days the loan will be paid up.

Q

17. 100 stones are placed on a straight road at intervals of 5 yards apart. A runner has to start from a basket placed 5 yards from the first stone, pick up the stones brings them back to the basket one by one. How many yards, has he to run altogether?

Solution: The man has to run 2 × 5 yard, 2 × 10 yard, 2 × 15 yards... for the first, second, 3[rd]... stones respectively.

∴ Total distance = 10 + 20 + 30 + ... to 100 terms

i.e. $S_{100} = 10 + 20 + 30 + \ldots$ to 100 terms

We have

Here $n = 100$

$a = 10$

$d = 20 - 10 = 10$

$$S_{100} = \frac{100}{2}[2(10)+(100-1)(10)] \cdots \left[S_n = \frac{n}{2}[2a+(n-1)\,d]\right] \quad \text{(formula)}$$

$$50\,[20+990]$$

$$S_{100} = 50{,}500$$

∴ The man has to run 50,500 yards altogether.

Q

18. A man saved Rs. 8250 in 10 months. In each month after the first month he saved Rs. 50 more than he did in the previous month. How much did he save in the first month?

Solution: Given $S_{10} = 8250$

Let Rs. x be the amount he saved in the first month.

∴ $x + (x + 50) + (x + 50 + 50) + \ldots$ ten terms = Rs. 8250

$x + x + 50 + x + 100 + \ldots$ ten terms = Rs. 8250

This is A.P. with $a = x$, $d = 50$, $S_{10} = 8250$ and $n = 10$.

We have

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$8250 = \frac{10}{2}[2x+(10-1)50]$$

$$8250 = 5\,[2x + 450]$$

$$\frac{8250}{5} = 2x + 450$$

$$1650 - 450 = 2x$$

$$1200 = 2x$$

$$\Rightarrow x = 600.$$

So the man saved Rs 600 in the first month.

19. A company manufactures 2400 cars in the first year. If the production capacity increases every year by 200, find in how many years the production will be doubled?

Given:

$$T_1 = 2400 = a$$

$$d = 200$$

$$T_n = 2400 \times 2 = 4800$$

$$n = ?$$

We have

$$T_n = a + (n-1)\,d$$

$$4800 = 2400 + (n-1)\ 200$$

$$4800 - 2400 = (n-1)\ 200$$

$$\frac{2400}{200} = n - 1$$

$$12 = n - 1$$

$$\Rightarrow \quad n = 12 + 1$$

$$n = 13.$$

The production will be doubled during the 13th year or after 12 years.

20. Firm X begins production with 1000 toys per year and decreases its production by 100 toys per year. Firm Y starts with 50 toys and raises production by 25 toys per year

(i) when will be the production of firms X and Y are equal?

(ii) What will be the production in that year?

Solution: Production of toys by Firm X

1000, 900, 800, ...

Production of toys by Firm Y:

500, 525, 550, ...

Let in the n^{th} year both X and Y produce same number of toys.

$$T_n \text{ for } X: a+(n-1)d$$

$$a = 1000,\ d = 900 - 1000 = -100$$

$$1000 + (n-1)(-100) \qquad \text{...(1)}$$

$$T_n \text{ for } Y = a+(n-1)d$$

$$a = 500,\ d = 525 - 500 = 25$$

$$= 500 + (n-1)25 \qquad \text{...(2)}$$

$$(1) = (2)$$

$$1000+(n-1)(-100)=500+(n-1)\,25$$
$$1000-100n+100=500+25n-25$$
$$1100-100n=475+25n$$
$$-100n-25n=475-1100$$
$$-125n=-625$$
$$n=\frac{-625}{-125}$$
$$n=5$$

$\therefore$ during the 5th year of production, the productions of firms X and Y are equal.

The production in that year: T_5

$$T_5 = a + (5-1)\,d$$
$$= a + 4d$$
$$= 500 + 4\,(25)$$
$$= 500 + 100$$
$$= 600$$

21. Amit buys saving certificate of value exceeding of the last years purchase by Rs. 100. After 10 years, he finds that the total value of the certificate purchased is Rs. 5000. Find the value of the certificate purchased by Amit in the (1) 1st year (2) in 7th year.

Solution: Given:

$$n = 10 \text{ yrs.}$$
$$d = 100$$
$$S_n = 5000$$
$$a = ?$$
$$T_n = ?$$

Formula:

$$S_n = \frac{n}{2}\left[2a+(n-1)\,d\right]$$
$$5000 = \frac{10}{2}\left[2a+(10-1)100\right]$$
$$5000 = 5\,[2a + 900]$$
$$1000 = 2a + 900$$
$$\Rightarrow \quad 2a = 100$$
$$a = 50$$

$\therefore$ Amit purchased Rs. 50 saving certificate in the first year.

$$T_n = a + (n-1)\,d \qquad \text{Formula}$$
$$T_7 = 50 + (7-1)\,100$$

$$T_7 = 50 + 600$$
$$T_7 = 650$$

The investment made by Amit in the 7^{th} year = Rs. 650.

3.6 GEOMETRIC PROGRESSION (GP)

A geometric progression is a sequence in which the ratio between any two consecutive terms is constant. The constant is called the common ratio of the G.P.

Example: 1, 2, 4, 8, 16, ... is a G.P. with first term 1 and common ratio 2.

n^{th} Term or General Term of a G.P.

Let a be the first term and r be the common ratio. Then

First term: $T_1 = a = ar^0$

Second term: $T_2 = a.r = ar^{2-1}$

Third term: $T_3 = ar^2 = ar^{3-1}$

$T_4 = ar^3 = ar^{4-1}$

$\vdots \quad \vdots \quad \vdots$

n^{th} term $T_n = ar^{n-1}$

Sum of the First n Terms of a G.P.

Let a be the first term 'r' be the common ratio and S_n be the sum to n terms.

$$\therefore \quad S_n = a + ar + ar^2 + \ldots + ar^{n-1} \qquad \ldots(1)$$

Multiplying by r

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \qquad \ldots(2)$$

Subtracting (2) from (1)

$$S_n - rS_n = a - ar^n \qquad \text{[Remaining terms get cancelled]}$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{where } r \neq 1$$

If r = 1, then

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

becomes

$$S_n = a + a + a + \ldots n \text{ times.}$$

$$S_n = na.$$

3.7 SUM TO INFINITY OF AN INFINITE G.P.:

If $|r|<1$ sum to ∞ of a G.P. $a+ar+ar^2+\ldots\infty$ is given by

$$S_\infty = \frac{a}{1-r}$$

This formula is obtained by $S_n = \dfrac{a(1-r^n)}{1-r}$ $\because |r|<1, r^n$ is very very small as $n \to \infty$

$(\text{neglecting } r^n) \therefore S_\infty = \dfrac{a}{1-r}$.

3.8 GEOMETRIC MEANS

If $a, x_1\ x_2 \ldots x_n b$ are in geometric progression. Then $x_1\ x_2 \ldots x_n$ are n geometric means between a and b.

To find the geometric means we note that $a_1\ x_1\ x_2\ x_3 \ldots x_n b$ are $(n+2)$ terms of G.P. whose first term is a and $(n+2)^{\text{nd}}$ term is b i.e. $T_{n+2} = b$

$$ar^{n+2-1} = b$$

$$ar^{n+1} = b$$

$$r^{n+1} = \frac{b}{a}$$

After finding r, we find $x_1 = ar_1$, $x_2 = ar^2 \ldots x_n = ar^n$.

Geometric mean: If a, x, b are in geometric progression, then x is called geometric mean between a and b.

Since a, x, b are in geometric progression

$$\frac{\text{any term}}{\text{previous term}} = \text{constant}$$

$\therefore$

$$\frac{x}{a} = \frac{b}{x}$$

Cross multiplying

$$x^2 = ab$$

$$x = \sqrt{ab}$$

$\therefore$ The Geometric mean between a and b is $\sqrt{ab}$.

WORKED EXAMPLES

1. Find the 7th term and sum to 7 term of the G.P. 3, 6, 12, ...

Given $a = 3$, $r = \frac{6}{3} = 2$

To find T_7 and S_7

We have

$$T_n = ar^{n-1}$$

$$T_7 = 3 \cdot 2^{7-1} = 3 \times 2^6 = 192$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_7 = \frac{3(1-2^7)}{1-2} = \frac{3(2^7-1)}{2-1}$$

$$S_7 = \frac{3(128-1)}{1} = 381$$

$$S_7 = 381$$

2. If the 5th term of GP is 81 and 2nd term is 24. Find the G.P.

Solution: Let the required G.P. be

$$a, ar, ar^2, ar^3 \ldots$$

$$n^{th} \text{ term} = T_n = ar^{n-1}$$

Given:

$$T_5 = 81 = ar^{5-1}$$

$$\therefore \quad ar^4 = 81 \qquad \ldots(1)$$

Also given

$$T_2 = 24$$

$$ar^{2-1} = 24$$

$$ar = 24 \qquad \ldots(2)$$

To get r,

$$Eq^n(1) \div Eq^n(2)$$

$$\frac{ar^4}{ar} = \frac{81}{24}$$

$$r^3 = \frac{27}{8}$$

$$r = \left(\frac{27}{8}\right)^{1/3} = \frac{3}{2}$$

Now from (2),

$$ar = 24$$

$$a \cdot \frac{3}{2} = 24$$

$$3a = 48$$

$$a = 16$$

$\therefore$ The required G.P.

$$= a, ar, ar^2 \ldots$$

$$16, 16\times\frac{3}{2}, 16\times\left(\frac{3}{2}\right)^2, 16\times\left(\frac{3}{2}\right)^4 \ldots$$

$$16, 24, 36, 54, \ldots$$

3. In a G.P. the common ratio is 2 and the sum of first 10 terms is 2046. Find the first term.

Solution: Let 'a' be the first term and 'r' be the common ratio.

Given:

$$r = 2$$

$$S_{10} = 2046$$

$$a = ?$$

Formula:

$$n = 10$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$2046 = \frac{a(1-2^{10})}{1-2}$$

$$2046 = \frac{a(2^{10}-1)}{2-1}$$

$$2046 = \frac{a(1024-1)}{1}$$

$$2046 = a(1023)$$

$$\Rightarrow \quad a = \frac{2046}{1023} = 2$$

$$a = 2$$

4. Find 3 numbers in G.P. Whose sum is 26 and their product is 216.

Solution: Let the 3 numbers in G.P. be

$$\frac{x}{r}, x, xr$$

Given: Their product = 216

$$\frac{x}{r} \times x \times xr = 216$$

$$x^3 = 216$$

$$x = 6$$

Given their sum = 26

$$\frac{x}{r} + x + xr = 26$$

$$\frac{6 + 6r + 6r^2}{r} = 26$$

$$6 + 6r + 6r^2 = 26r$$

$$6r^2 + 6r - 26r + 6 = 0$$

$$6r^2 - 18r - 2r + 6 = 0$$

$$6r(r-3) - 2(r-3) = 0$$

$$r = 3 \text{ or } r = \frac{1}{3}$$

$\therefore$ The numbers are $\frac{x}{r}, x, xr$

$$= \frac{6}{3}, 6, 6 \times 3$$

$$= 2, 6, 18$$

5. Find the sum to 'n' terms: 7 + 77 + 777 + ...

Let $$S = 7 + 77 + 777 + 7777 + \ldots \quad \ldots(1)$$

This is not a geometric progression. To convert it into G.P. Divide the equation (1) by 7 and multiply by 9.

$$\frac{S}{7} = 1 + 11 + 111 + \ldots$$

$$\frac{9S}{7} = 9 + 99 + 999 + \ldots$$

$$\frac{9S}{7} = (10-1) + (100-1) + (1000-1) + \ldots$$

$$\frac{9S}{7} = (10 + 100 + 1000 + \ldots) + (-1 - 1 - 1 - \ldots)$$

Now 10 + 100 + ... is a G.P. with $a = 10,\ r = \frac{100}{10} = 10,,$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{10(10^n - 1)}{9}$$

$$\therefore \quad \frac{9S}{7} = \frac{10(10^n - 1)}{9} - n$$

$$S = \frac{7}{9}\left[\frac{10(10^n - 1)}{9} - n\right]$$

$$S = \frac{70(10^n - 1)}{81} - \frac{7n}{9}$$

6. If a, b, c and d are in G.P. then prove that $a + b$, $b + c$, $c + d$ are in G.P.

Proof:

Given a, b, c, d are in G.P.

So $$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = r \text{ (common ratio)}$$

$$b = ar,\ c = br,\ d = cr$$

Now

$$a + b = a + ar = a\,(1 + r)$$

$$b + c = ar + br = ar + ar^2 = ar\,(1 + r)$$

$$c + d = br + cr = ar^2 + ar^3 = ar^2\,(1 + r)$$

Again

$$\frac{b+c}{a+b} = \frac{ar(1+r)}{a(1+r)} = r$$

$$\frac{c+d}{b+c} = \frac{ar^2(1+r)}{ar(1+r)} = r$$

$\therefore$ $a + b$, $b + c$, $c + d$ are in G.P. with common ratio r.

7. If S_1, S_2 and S_3 are the sums of n, $2n$ and $3n$ terms respectively of a same G.P. then

Prove that

$$S_1(S_3 - S_2) = (S_2 - S_1)^2$$

Proof: Let a be the first term, r be the common ratio. Then sum of first n terms of GP

$$S_n = \frac{a(1-r^n)}{1-r} = S_1 \quad \text{(Given)}$$

Similarly Sum of first $2n$ terms $= S_{2n} = \dfrac{a(1-r^{2n})}{1-r} = S_2$

Sum of first $3n$ terms $= S_{3n} = \dfrac{a(1-r^{3n})}{1-r} = S_3$

To prove:

$$S_1(S_3 - S_2) = (S_2 - S_1)^2$$

Consider LHS:

$$S_1(S_3 - S_2)$$

$$= \frac{a(1-r^n)}{1-r}\left[\frac{a(1-r^{3n})}{1-r} - \frac{a(1-r^{2n})}{1-r}\right]$$

$$= \frac{a(1-r^n)}{(1-r)}\left[\frac{a(1-r^{3n}-1+r^{2n})}{1-r}\right]$$

$$= \frac{a^2(1-r^n)(r^{2n}-r^{3n})}{(1-r)^2}$$

$$= \frac{a^2(r^{2n}-r^{3n}-r^{3n}+r^{4n})}{(1-r)^2}$$

$$= \frac{a^2(r^{2n}-2r^{3n}+r^{4n})}{(1-r)^2}$$

Now RHS:

$$(S_2-S_1)^2 = \left[\frac{a(1-r^{2n})}{1-r} - \frac{a(1-r^n)}{1-r}\right]^2$$

$$= \left[\frac{a(1-r^{2n})-a(1-r^n)}{1-r}\right]^2$$

$$= \frac{a^2(1-r^{2n}-1+r^n)^2}{(1-r)^2}$$

$$\frac{a^2(r^n-r^{2n})^2}{(1-r)^2} = \frac{a^2(r^{2n}+r^{4n}-2r^2r^{2n})}{(1-r)^2}$$

$$(S_2-S_1)^2 = \frac{a^2(r^{2n}-2r^{3n}+r^{4n})}{(1-r)^2} \quad ...(2)$$

Hence (1) = (2)

LHS = RHS.

8. Insert 3 geometric means between 5 and 405.

Solution: Let the required G.P. be

$5, x_1, x_2, x_3, 405$

Here $a = 5$, 5^{th} term $= 405$

But $T_n = ar^{n-1}$

$$405 = 5r^{5-1}$$

$$\frac{405}{5} = r^4$$

$$81 = r^4 \Rightarrow r = 3$$

$\therefore$

$$x_1 = ar = 5 \times 3 = 15$$

$$x_2 = ar^2 = 15 \times 3 = 45$$

$$x_3 = ar^3 = 45 \times 3 = 135$$

$\therefore$ Required G.P. is 5, 15, 45, 135, 405.

3.9 APPLICATION OF GEOMETRIC PROGRESSION TO BUSINESS PROBLEMS

9. Sriram borrows Rs. 32,760 without interest and agrees to pay back in 12 monthly instalments, each installment being twice the preceding one. Find the 3^{rd} and last installment.

Solution: Let the first installment be Rs. a

Then 2^{nd} installment $= 2a$

3^{rd} installment $= 2\,(2a) = 4a$

Clearly the instalments

$a, 2a, 4a, \ldots$ form a G.P. with first term $= a$ and common ratio $= 2$

Sum of 12 instalments = Rs. 32,760

i.e. $S_{12} = 32760$

But

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{12} = \frac{a(2^{12} - 1)}{2 - 1}$$

$$32760 = a(4096 - 1)$$

$$= 8000 \times (0.8)^4$$

$$= 8000 \times 0.4096$$

$$= \text{Rs. } 3276.80$$

11. X, who owes his partner Y a sum of Rs. 4000 in a business transaction, is requested by Y to pay the amount in 12 daily installments. Commencing with one paise a day and twice the amount on each successive day. Considering this as a very profitable and easy device of clearing the debt, X accepts it. Do you think that X will gain in the bargain? Find X's gain or loss.

Solution: The daily installments,

1 paise, 2 paise, 4 paise, 8 paise.......

form a G.P. with $a = 1$, $r = 2$ and $n = 12$.

We have

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{1[2^{12} - 1]}{2 - 1} = 2^{12} - 1$$

$$S_n = 4096 - 1 = 4095$$

Hence totally X pays Rs. 4095.

So X loses in the bargain.

Loss = Rs. 4095 – Rs. 4000 = Rs. 95

12. A person is entitled to receive an annual payment which for each year is less by one-tenth of what it was for the year before. If the first payment is Rs. 100, Prove that he cannot receive more than Rs. 1000 however long he may live.

Solution:

First installment = Rs. 100

2nd installment = 100 – one tenth of first installment

$$100 - \frac{1}{10} \times 100$$

$$100 - 10 = \text{Rs. } 90.$$

$$\text{3rd installment} = 90 - \frac{1}{10} \times 90$$

$$= 90 - 9 = 81$$

Now, 100, 90, 81, ... is a G.P.

With $a = 100,\ r = \frac{90}{100} = \frac{9}{10}$.

Sum to $\infty = S_\infty = \frac{a}{1-r}$

$$S_\infty = \frac{100}{1-\frac{9}{10}} = \frac{100}{\frac{1}{10}} = 1000$$

$\therefore$ Maximum amount that the person receives however long he may live is Rs. 1000.

13. A golf ball is thrown vertically upwards to a height of 100 mts. After striking the ground it rebounds three fifths of the height to which it rose first and so on for successive rebounds find the total distance covered by the ball before it stops.

Solution: Before the first rebound, the ball has covered $100 \times 2 = 200$ mts.

After the first rebound, the ball will rise $\frac{3}{5} \times 100$ m and covers $2 \times \frac{3}{5} \times 100$ mts. before the 2nd rebound. The ball will stop after infinite reboundings.

$\therefore$ Total distance covered

$$= 2\times 100 + 2\times 100 \times \frac{3}{5} + 2\times 100 \times \left(\frac{3}{5}\right)^2 + \ldots \infty$$

$$2\times 100\left[1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \ldots \infty\right]$$

$1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \ldots \infty$ is an infinite G.P.

with $a = 1,\ r = \frac{3}{5}$.

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{3}{5}}$$

$$= \frac{1}{\frac{2}{5}} = \frac{5}{2}.$$

$$2\times 100 \times \frac{5}{2}$$

$$= 500.$$

$\therefore$ Total distance covered = 500 mts.

14. India's population is increasing by 2.1% per year. Population in 1950 was 36 crores. Find the population in 1960.

Solution: Population in 1950 = 36 crores.

$$= \frac{H-c+H-a}{(H-a)(H-c)} = \frac{2H-c-a}{(H-a)(H-c)}$$

$$= \frac{2\left[\frac{2ac}{a+c}\right]-c-a}{\left(\frac{2ac}{a+c}-a\right)\left(\frac{2ac}{a+c}-c\right)} = \frac{\frac{4ac-ac-c^2-a^2-ac}{a+c}}{\frac{(2ac-a^2-ac)(2ac-ac-c^2)}{(a+c)^2}}$$

$$\frac{-(a+c)(a-c)^2}{-ac(a-c)(a-c)} = \frac{a+c}{ac} = \frac{1}{c}+\frac{1}{a}$$

$$= \frac{1}{a}+\frac{1}{c} = \text{RHS.}$$

Hence proved.

MISCELLANEOUS PROBLEMS

1. The 3 numbers are in the ratio 3 : 7 : 9. If 5 is subtracted from the second, the resulting numbers form an A.P. Find the original numbers?

Solution: Given 3 numbers are in ratio 3 : 7 : 9.

Let 3x, 7x and 9x be the numbers. If 5 is subtracted from second the resulting number 3x, 7x − 5, 9x are in A.P.

In an A.P.,

Any term − previous term = constant.

$$\therefore \quad 7x-5-3x = 9x-(7x-5)$$

$$4x-5 = 9x-7x+5$$

$$4x-5 = 2x+5$$

$$4x-2x = 5+5$$

$$2x = 10$$

$$x = 5$$

Hence the original numbers are 3x, 7x and 9x.

3 (5), 7 (5) and 9 (5)

= 15, 35 and 45.

2. The last term of a series in A.P. is 40, the sum of the series is 952 and common difference is −2. Find the 1st term the number of terms.

Given: $l = T_n = 40$

$S_n = 952$

$d = -2$

$a = ?$

$n = ?$

We have
$$T_n = a + (n-1)\,d$$
$$40 = a + (n-1)(-2)$$
$$40 = a - 2n + 2$$
$$40 - 2 = a - 2n$$
$$38 = a - 2n$$
$$\Rightarrow \quad a = 38 + 2n \qquad \ldots(1)$$

Now
$$S_n = \frac{n}{2}[a + T_n]$$
$$952 = \frac{n}{2}[a + 40]$$

Substituting $a = 38 + 2n$, from (1)
$$952 = \frac{n}{2}[38 + 2n + 40]$$
$$952 = \frac{n}{2}[78 + 2n]$$
$$952 = 39n + n^2$$
$$n^2 + 39n - 952 = 0$$

$-952\,n^2 \Big<\begin{array}{l} +56n \\ -17n \\ \hline +39n \end{array}$

$$n^2 + 56n - 17n - 952 = 0$$
$$n(n + 56) - 17(n + 56) = 0$$
$$n = -56 \quad \text{or} \quad n = 17$$

$\because$ n is non negative.

From (1), $n = 17.$

$$a = 38 + 3n$$
$$a = 38 + 2\,(17)$$

$a = 38 + 34$

$a = 72.$

1st term = 72 and number of terms = 17.

3. The cost of a drilling a tube well is Rs. 2.50 per foot for the first 100 feet and an additional Re. 0.25 for every subsequent foot. Find the cost of the last foot and the total cost of 220 feet deep tube well.

Solution: Cost of drilling a 100 feet well

$= 100 \times \text{Rs. } 2.50 = \text{Rs. } 250$

An additional cost Re. 0.25 is charged for every subsequent foot.

For 101st feet it is Rs. 2.75, for 102nd feet it is Rs. 3.00 and so on.

Cost of for remaining (220 – 100 = 120)

feet are

2.75, 3.00, 3.25, ... which form an A.P. with $a = 2.75$, $d = 0.25$, $n = 120$.

$$S_n = \frac{n}{2}[2a(n-1)d]$$

$$S_{120} = \frac{120}{2}[2(2.75)+(120-1)(0.25)]$$

$$= 60[5.50+(119)(0.25)]$$

$$= 60[5.50+29.75]$$

$$S_{120} = 2115.$$

$\therefore$ Total cost of tubewell of 220 ft deep

$= 2115 + 250 = \text{Rs. } 2365$

Cost of last foot $= T_{120}$

$$T_n = a + (n-1)d$$

$$T_{120} = 2.75+(120-1)(0.25)$$

$$= 2.75+(119)(0.25)$$

$$= 2.75+29.75$$

$$T_{120} = 32.50$$

$\therefore$ Cost of last foot = Rs. 32.50.

4. The 3rd term of a G.P. is 12 and the 6th term is 96. Find the sum of first 9 terms.

Solution: Let a be the first term and r be the common ratio of the G.P.

We have n^{th} term: $T_n = ar^{n-1}$

Sum of 1st n terms: $S_n = \dfrac{a(1-r^n)}{1-r}$

Given 3rd term = 12.

$$T_3 = ar^{3-1} = 12$$

$$ar^2 = 12 \quad \text{...(1)}$$

6th term = 96

$$T_6 = ar^{6-1} = 96$$

$$ar^5 = 96 \quad \text{...(2)}$$

Dividing (2) by (1) to get the value of r.

$$\frac{ar^5}{ar^2} = \frac{96}{12}$$

$$r^3 = 8$$

$$\Rightarrow \quad r = 2.$$

From (1)

$$ar^2 = 12$$

$$a\,(2)^2 = 12$$

$$\Rightarrow \quad a = \frac{12}{4} = 3$$

Now sum of the 1st 9 terms

$$S_9 = \frac{a(1-r^9)}{1-r}$$

$$S_9 = \frac{3(1-2^9)}{1-2}$$

$$= \frac{3(2^9-1)}{2-1} = 3(512-1) = 1533$$

5. 3 numbers whose sum is 18 are in A.P. If 2, 4 and 11 are added to them respectively, the resulting series are in G.P. Find the numbers:

Solution: Let the 3 numbers in A.P. be

$$x-d,\ x,\ x+d.$$

Given their sum = 18.

$$x-d+x+x+d=18$$
$$3x=18$$
$$x=6$$

If 2, 4 and 11 are added to the numbers $x-d,\ x,\ x+d$, we get

$$x-d+2,\ x+4,\ x+4+11$$

Given these numbers are in G.P.

In G.P.,

$$\frac{\text{any term}}{\text{previous term}}=\text{constant}$$

$$\therefore \quad \frac{T_2}{T_1}=\frac{T_3}{T_2}$$

$$\frac{x+4}{x-d+2}=\frac{x+d+11}{x+4}$$

Also $x = 6$.

$$\frac{6+4}{6-d+2}=\frac{6+d+11}{6+4}$$

$$\frac{10}{8-d}=\frac{17+d}{10}$$

$$100=(8-d)(17+d)$$

$$100=136+8d-17d-d^2$$

$$d^2+9d-36=0$$

$$d^2+12d-3d-36=0$$

$$d(d+12)-3(d+12)=0$$

$$d=+3 \text{ or } d=-12$$

When $d = 3$,

The number are

$x-d,\ x,\ x+d$

$6-3, 6, 6+3$

$3, 6, 9.$

When $d = 12$,

The numbers are

$x-d,\ x,\ x+d$

$6-(-12), 6, 6+(-12)$

$6+12, 6, 6-12$

$18, 6, -6.$

$$= \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

$$\therefore 1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots\infty = 24$$

10. The distance passed over by a certain pendulum bob in succeeding swings form a G.P. 16, 12, 9... cm respectively. Calculate the distance traversed by the bob before it comes to rest.

Solution: The distance traversed by the bob before it comes to rest

$$= 16 + 12 + 9 + \ldots \infty$$

This is an infinite G.P. with

$$a = 16, r = \frac{12}{16} = \frac{3}{4}$$

$$S_\infty = \frac{a}{1-r} \text{ [formula]}$$

$$S_\infty = \frac{16}{1-\frac{3}{4}} = \frac{16}{\frac{1}{4}} = 64.$$

$\therefore$ The distance traversed by the bob = 64 cm

11. The sum of the digits of a 3 digit number is 12. The digits are in A.P. If the digits are reversed, then the number is increased by 396. Find the number.

Solution: Let the 3 digit number be xyz.

Given: The digits are in A.P.

So,

$$x = a - d,\ y = a \text{ and } z = a + d$$

Given: Sum of the digits = 12.

$$x + y + z = 12$$
$$a - d + a + a + d = 12$$
$$3a = 12$$
$$a = 4$$

Now, any 3 digit number say 657 can be written as

$$657 = 600 + 50 + 7$$
$$= 6(100) + 5(10) + 7$$

Similarly 3 digit number xyz can be written as $xyz = x(100) + y(10) + z$

If the digits, xyz are reversed, we get zyx.

Again zyx can be written as

$$zyx = z(100) + y(10) + x$$

Given: $zyx - xyz = 396$

$[z(100) + y(10) + x] - [x(100) + y(10) + z] = 396$

$100z + 10y + x - 100x - 10y - z = 396.$

$99z - 99x = 396$

But $x = a - d,\ z = a + d$ and $a = 4$

$99(4 + d) - 99[4 - d] = 396.$

$99(4 + d - 4 + d) = 396$

$99(2d) = 396$

$$d = \frac{396}{99(2)} = \frac{398}{198} = 2.$$

Hence the digits are xyz

$a - d,\ a,\ a + d$

$4 - 2,\ 4,\ 4 + 2$

$2, 4, 6.$

∴ The required number is 246.

12. If a, b, c are in G.P. then Prove that $\log_a n, \log_b n, \log_c n$ are in H.P.

Proof: Given: a, b, c are in G.P.

⇒ $b^2 = ac$

taking log to base n on both sides

$$\log_n b^2 = \log_n (ac)$$

⇒ $$2\log_n b = \log_n a + \log_n c$$

⇒ $$\log_n b = \frac{\log_n a + \log_n c}{2}$$

⇒ $\log_n a, \log_n b, \log_n c$ are in A.P.

⇒ $\frac{1}{\log_n a}, \frac{1}{\log_n b}, \frac{1}{\log_n c}$ are in H.P.

i.e., $\log_a n, \log_b n, \log_n c$ are in H.P.

13. If a, b, c are in A.P., b, c, d are in G.P., and c, d, e are in H.P. then prove that $c^2 = ae$.

Proof: a, b, c are in A.P., $b = \frac{a+c}{2}$ ...(1)

b, c, d are in G.P., $c^2 = bd$ ...(2)